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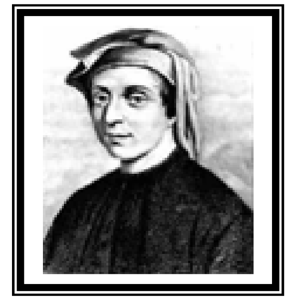
M111
1/10/99

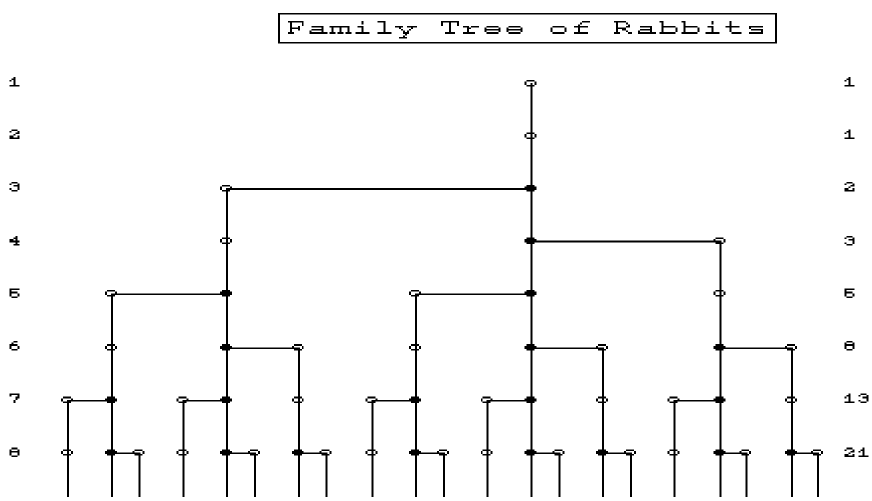
Leonardo Pisano Fibonacci (c.1175 - c.1240)



Fibonacci, also known as Leonardo of Pisa, was the greatest European mathematician of the Middle Ages. He was born in Pisa, Italy 1175 AD.

Pisa was an important commercial town in its day and had links with many Mediterranean ports. Leonardo's father was a kind of customs officer in the North African town of Bugia. So Leonardo grew up with a North African education under the Moors and later traveled extensively around the Mediterranean coast. He would have met with many merchants and learned of their systems of doing arithmetic. He soon realized the many advantages of the "Hindu-Arabic" system over all the others.

One day in 1202 while Fibonacci sit watching rabbits play in his fathers field he came up with the realization that if a female rabbit gave birth to one pair of rabbits, there would be a total of three rabbits. If another pair of rabbits were born there would be a total of five rabbit. And if another pair was born there would be a total of eight rabbits, and so on and so on. This discovery became known as the Fibonacci Sequence.



In more simple terms:

0, 1, 1, 2, 3, 5, 8, 13, . . . (add the last two to get the rest)

Fibonacci and the Golden Number

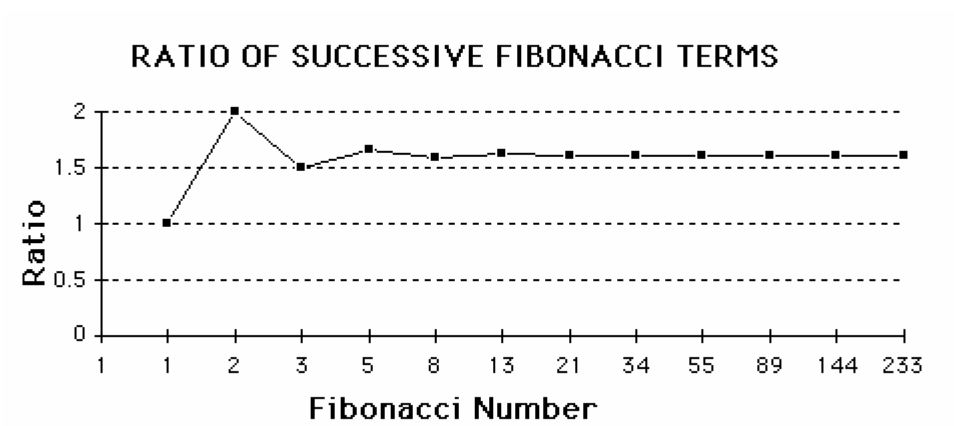
The Fibonacci numbers are defined as such:

F(1) = 1
F(2) = 1
F(3) = 2
F(4) = 3
F(5) = 5
F(6) = 8
F(7) = 13
F(8) = 21
F(9) = 34
F(10) = 55

If we take the ratio of two successive numbers in Fibonacci's series, (1, 1, 2, 3, 5, 8, 13, . . .) and we divide each by the number before it, we will find the following series of numbers:

$^{1}/_{1}$ = 1
$^{2}/_{1}$ = 2
$^{3}/_{2}$ = 1.5
$^{5}/_{3}$ = 1.666
$^{8}/_{5}$ = 1·6
$^{13}/_{8}$ = 1.625
$^{21}/_{13}$ = 1.61538

It is easier to see what is happening if we plot the ratios on a graph:

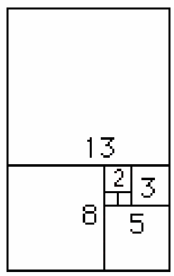


The ratio seems to be settling down to a particular value, which we call the *golden ratio* or the *golden number*. It has a value of approximately 1·61804, although there is an even more accurate value.

The *golden ratio* 1·618034 is also called the *golden section* or the *golden mean* or just the *golden number*. It is often represented by the Greek letter *Phi* . The closely related value which we write as *phi* (with a small "p"), is just the decimal part of Phi, namely 0·618034.

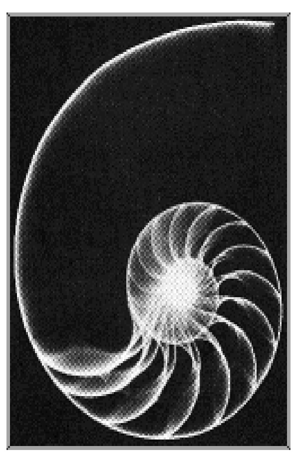
The Fibonacci Rectangles and Shell Spirals

Another example of the Fibonacci numbers is by starting with two small squares of size 1 next to each other. Then drawing a square of size 2 (=1+1). Now draw a new square - touching both a unit square and the latest square of side 2 - so having sides 3 units long; and then another touching both the 2-square and the 3-square (which has sides of 5 units). Then continue adding squares around the picture, each new square having a side, which is as long as the sum of the latest two square's sides. This set of rectangles whose sides are two successive Fibonacci numbers in length and which are composed of squares with sides, which are Fibonacci numbers, this is called the **Fibonacci Rectangles**.



The **Fibonacci Spiral** is another example. A similar curve to this occurs in nature as the shape of a snail shell or some seashells. Whereas the Fibonacci Rectangles spiral increases in size by a factor of Phi (1.618…) in a *quarter of a turn* (i.e. a point a further quarter of a turn round the curve is 1.618... times as far from the center, and this applies to *all* points on the curve), the Nautilus spiral curve takes a *whole turn* before points move a factor of 1.618... from the center.

These spiral shapes are called Equiangular or Logarithmic spirals. The links from these terms contain much more information on these curves and pictures of computer-generated shells.



Fibonacci Patterns

Here are some patterns of the Fibonacci numbers that have been discovered:

- There is a cycle in the **units** column - the cycle of units digits (0,1,1,2,3,5,8,3,1,4, …) repeats from n=60 and again every 60 values.
- There is also a cycle in the **last two** digits, repeating (00, 01, 01, 02, 03, 05, 08, 13, ...) from n=300 with a cycle of length 300.
- For the last **three** digits, the cycle length is 1,500
- For the last **four** digits, the cycle length is 15,000 and
- For the last **five** digits the cycle length is 150,000 and so on...

The ***even*** Fibonacci numbers are:

F(3), F(6), F(9), F(12), ... i.e. F(3k)
Every third Fibonacci number is a multiple of 2

Those Fibonacci numbers, which are **multiples of 3**, are:

F(4), F(8), F(12), F(16), ... i.e. F(4k)
Every fourth Fibonacci number is a multiple of 3.

Every *fifth* Fibonacci number is a multiple of F(5)=5

Every *sixth* Fibonacci number is a multiple of F(6)=8

Therefore the general rule is:

Every kth Fibonacci number is a multiple of F(k)

or, expressed mathematically,

F(nk) is a multiple of F(k) for all values of n and k=1,2,...

This means that if the subscript is composite (not a prime) then so is that Fibonacci number, with one exception. Unfortunately, the converse is not always true (that is, it is not true that if a subscript is prime then so is that Fibonacci number).
The first case to show this is the 19th position (and 19 is prime) but F(19)=4181 and F(19) is *not* prime as 4181=113x37.

The Fibonacci Numbers in Pascal's Triangle

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

Each entry in the triangle on the left is the sum of the two numbers either side of it but in the row above. A blank space can be taken "0" so that each row starts and ends with "1".

$$Fib(n) = \sum_{k=0}^{n-1} \binom{n-k-1}{k}$$

It is easy to see that the diagonal sums really are the Fibonacci numbers if we remember that each number in Pascal's triangle is the sum of two numbers in the row above it (blank spaces count as zero), so that 6 here is the sum of the two 3's on the row above:

The numbers in any diagonal row are therefore formed from adding numbers in the previous two diagonal rows as we see here where all the blank spaces are zeroes and where we have introduced an extra column of zeros, which we will use later:

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

Pascal		Fibonacci
0 + 1	=	1
1 + 1	=	2
1 + 2	=	3
1 + 3 + 1	=	5
1 + 4 + 3	=	8
1 + 5 + 6 + 1	=	13

The *sum* of the numbers on one diagonal is the sum of the numbers on the previous two diagonals.

If we let D(i) stand for the sum of the numbers on the Diagonal that starts with one of the extra zeros at the beginning of row i, then

D(0)=0 and D(1)=1

What is also shown that this is always true: one diagonals sum is the sum of the previous two diagonal sums, or, in terms of our D series of numbers:

$$D(i) = D(i-1) + D(i-2)$$

But...

$$D(0) = 1$$
$$D(1) = 1$$
$$D(i) = D(i-1) + D(i-2)$$

is exactly the definition of the Fibonacci numbers! So D(i) is just F(i) and the sums of the diagonals in Pascal's Triangle are the Fibonacci numbers!

We can see another arrangement of Pascal's triangle by drawing Pascal's Triangle with all the rows moved over by 1 place, we have a clearer arrangement, which shows the Fibonacci numbers as sums of columns:

0	1	2	3	4	5	6	7	8	9		
	0	1									
	1		1	1							
	2			1	2	1					
	3				1	3	3	1			
	4					1	4	6	4	1	
	5						1	5	10	10	5
	6							1	6	15	20
	7								1	7	21
	8									1	8
	9										1
		1	1	2	3	5	8	13	21	34	55

Fibonacci's Rabbit Generations and Pascal's Triangle

Here's another explanation of how the Pascal triangle numbers sum to give the Fibonacci numbers, this time explained in terms of the original rabbit problem.

For example, the sum of F(8) = 21

0	1	2	3	4	5	6	7	8	9	
	0	1								
	1	1	1							
	2	1	2	1						
	3	1	3	3	1					
	4	1	4	6	4	1				
	5	1	5	10	10	5	1			
	6	1	6	15	20	15	6	1		
	7	1	7	21	35	35	21	7	1	
	8	1	8	28	56	70	56	28	8	1

The general pattern for month n and level (generation) k is
Level k: is Pascal's triangle row n-k and column k-1 For month n we sum all the generations as k goes from 1 to n (but half of these will be zeros).

The Fibonacci Series as a Decimal Fraction

Have a look at this decimal fraction:

0·0112359550561...

It looks like it begins with the Fibonacci numbers, 0, 1, 1, 2, 3 and 5 and indeed it does if we express it as:

```
0·0
  1
   1
    2
     3
      5
       8
       13
        21
         34
          55
           89
           144
             ...
--------------------
0·011235955056179...
```

The value of this decimal fraction can be expressed as:

$$0/10 + 1/100 + 1/1000 + 2/10^4 + 3/10^5 + ...$$

or, using powers of 10 and replacing the Fibonacci numbers by F(i):

$$F(0)/10^1 + F(1)/10^2 + F(2)/10^3 + ... + F(n-1)/10^n + ...$$

or, if we use the negative powers of 10 to indicate the decimal fractions:

$$F(0)10^{-1} + F(1)10^{-2} + F(2)10^{-3} + ... + F(n-1)10^{-n} + ...$$

To find the value of the decimal fraction we look at a generalization, replacing 10 by **x**.

Let P(x) be the polynomial in x whose coefficients are the Fibonacci numbers:

$$P(x)= 0 + 1 x^2 + 1 x^3 + 2x^4 + 3x^5 + 5x^6 + ...$$

$$\text{or } P(x)=F(0)x + F(1)x^2 + F(2)x^3 + ...+F(n-1)x^n + ...$$

To avoid confusion between the variable x and the multiplication sign x, we will represent multiplication by *:

The decimal fraction 0.011235955... above is just

$$0*(1/10) + 1*(1/10)^2 + 1*(1/10)^3 + 2*(1/10)^4 + 3*(1/10)^5 + ... + F(n-1)*(1/10)^n + ...$$

Which is just P(x) with x taking the value (1/10), which we write as P(1/10).

Now here is the interesting part of the technique!

We now write down xP(x) and $x^2P(x)$ because these will "move the Fibonacci coefficients along":

$$P(x)=F(0)x + F(1)x^2 + F(2)x^3 + F(3)x^4 + ... +F(n-1)x^n + ...$$

$$xP(x)=F(0)x^2 + F(1)x^3 + F(2)x^4 + ... +F(n-2)x^n + ...$$

$$x^2P(x)=F(0)x^3 + F(1)x^4 + ... +F(n-3)x^n + ...$$

We can align these terms up so that all the same powers of x are in the same column (as we would do when doing ordinary decimal arithmetic on numbers) as follows:

$$\begin{array}{llllll} P(x)= & F(0)x + & F(1)x^2 + & F(2)x^3 + & F(3)x^4 + ... & +F(n-1)x^n + ... \\ xP(x)= & & F(0)x^2 + & F(1)x^3 + & F(2)x^4 + ... & +F(n-2)x^n + ... \\ x^2P(x)= & & & F(0)x^3 + & F(1)x^4 + ... & +F(n-3)x^n + ... \end{array}$$

We have done this so that each Fibonacci number in P(x) is aligned with the two previous Fibonacci numbers. Since the sum of the two previous numbers always equals the next in the Fibonacci series, then, when we take them away, the result will be zero - the terms will vanish!

So, if we take away the last two expressions (for $xP(x)$ and $x^2P(x)$) from the first equation for P(x), the right-hand side will simplify since all but the first few terms vanish, as shown here:

$$P(x)=F(0)x + F(1)x^2 + F(2)x^3 + F(3)x^4 + \ldots +F(n-1)x^n + \ldots$$

$$xP(x)= \quad F(0)x^2 + F(1)x^3 + F(2)x^4 + \ldots +F(n-2)x^n + \ldots$$

$$x^2P(x)= \quad F(0)x^3 + F(1)x^4 + \ldots +F(n-3)x^n + \ldots$$

$$(1-x-x^2)P(x)=F(0)x +(F(1)-F(0))x^2 + (F(2)-F(1)-F(0))x^3+\ldots$$

Apart from the first two terms, the general term, which is just the coefficient of x^n, becomes F(n)-F(n-1)-F(n-2) and, since F(n)=F(n-1)+F(n-2) all but the first two terms become zero which is why we wrote down $xP(x)$ and $x^2P(x)$:

$$(1-x-x^2)P(x) = 0+1x^2+0x^3+0x^4+..+0x^n+..$$

$$(1-x-x^2)P(x) = x^2$$

So,

$$P(x)= \frac{x^2}{1 - x - x^2} = \frac{1}{x^{-2} - x^{-1} -1}$$

Now our fraction is just P(1/10), and the right hand side tells us its exact value:

1 / (100-10-1)=**1/89**=0·0112358...

From our expression for P(x) we can also deduce the following:

10/89=0·112359550561...

If x=1/100, we have

P(1/100)=0·00 01 01 02 03 05 08 13 21 34 55 ...
= 1/(10000-100-1) = 1/9899

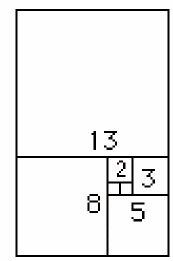
and
100/9899=0·0101020305081321345 5...

and so on.

By looking at the Fibonacci squares and spiral again, we see that wherever we stop, we will always get a rectangle, since the next square to add is determined by the longest edge on the current rectangle. Also, the longest edges are just the sum of the latest two sides-of-squares to be added. Since we start with squares of sides 1 and 1, this tells us why the square sides are the Fibonacci numbers (the next is the sum of the previous 2).

Also, we see that each rectangle is a jigsaw puzzle made up of all the earlier squares to form a rectangle. All the squares and all the rectangles have sides, which are Fibonacci numbers in length. We express each rectangle's area as a sum of its component square areas:

The diagram shows that



$$1^2+1^2+2^2+3^2+5^2+8^2+13^2=13x21$$

and also, the smaller rectangles show:

$$1^2+1^2=1x2$$
$$1^2+1^2+2^2=2x3$$
$$1^2+1^2+2^2+3^2=3x5$$
$$1^2+1^2+2^2+3^2+5^2=5x8$$
$$1^2+1^2+2^2+3^2+5^2+8^2=8x13$$

This picture actually is a convincing proof that the pattern will work for any number of squares of Fibonacci numbers that we wish to sum. They always total to the largest Fibonacci number used in the squares multiplied by the next Fibonacci number.

That is a bit of a mouthful to say - and to understand - so it is better to express the relationship in the language of mathematics:

$$1^2 + 1^2 + 2^2 + 3^2 + ... + F(n)^2 = F(n)F(n+1)$$

and it is true for ANY n from 1 upwards.

Binet's Formula for the nth Fibonacci number

We have only defined the nth Fibonacci number in terms of the two before it. The n-th Fibonacci number is the sum of the (n-1)th and the (n-2)th. So to calculate the 100th Fibonacci number, for instance, we need to compute all the 99 values before it first - quite a task, even with a calculator!

A natural question to ask therefore is:

Can we find a formula for F(n), which involves only n and does not need any other (earlier) Fibonacci values?

Yes! It involves our golden section number Phi and its reciprocal phi.

Here it is:

$$\text{Fib}(n) = \frac{\text{Phi}^n - (-\text{Phi})^{-n}}{\text{sqrt}(5)}$$

If you don't like negative powers then we can use $x^{-n}=(1/x)^n$:

$$\text{Fib}(n) = \frac{\text{Phi}^n - (-1/\text{Phi})^n}{\text{sqrt}(5)}$$

If you also don't like taking reciprocals then we can use phi=1/Phi and the formula can be written as:

$$\text{Fib}(n) = \frac{\text{Phi}^n - (-\text{phi})^n}{\text{sqrt}(5)}$$

Here phi = 1/Phi and also phi = Phi-1 = (sqrt(5)-1)/2 = 0·6180339....
We can also write this in terms of square-root-of-5
(since Phi=(sqrt(5)+1)/2) and get:

$$\text{Fib}(n) = \frac{\Phi^n - (-\phi)^n}{\Phi - \phi} = \frac{\Phi^n - (-\phi)^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

If you prefer values in your formula, then here is another form:-

$$\text{Fib}(n) = \frac{1{\cdot}6180339..^n - (-0{\cdot}6180339..)^n}{\text{sqrt}(5)}$$

This is a surprising formula since it involves square roots and powers of Phi (an irrational number) but it always gives an *integer* for all (integer) values of n!

Here's how it works:

Let X= Phi^n =(1·618..)^n

Let Y=(-Phi)$^{-n}$=(-1·618..)$^{-n}$=(-0·618..)^n then we have:

n	X = Phi^n	Y = (-Phi)$^{-n}$	X-Y	(X-Y)/sqrt(5)
0	1	0	0	0
1	1.618033989	-0.61803399	2.23606798	1
2	2.618033989	0.38196601	2.23606798	1
3	4.236067977	-0.23606798	4.47213595	2
4	6.854101966	0.14589803	6.70820393	3
5	11.09016994	-0.09016994	11.18033989	5
6	17.94427191	0.05572809	17.88854382	8
7	29.03444185	-0.03444185	29.06888371	13
8	46.97871376	0.02128624	46.95742753	21
9	76.01315562	-0.01315562	76.02631123	34
10	122.9918694	0.00813062	122.9837388	55

You might want to look at two ways to prove this formula: the first way is very simple and the second is more advanced and is for those who are already familiar with matrices.

Since phi is less than one in size, its powers decrease rapidly. We can use this to derive the following simpler formula for the n-th Fibonacci number F(n):

F(n) = round(Phi^n / 5)

where the round function gives the nearest integer to its argument.

n	Phi^n/sqrt(5)	Rounded
0	0·447213595	0
1	0·723606798	1
2	1·170820393	1
3	1·894427191	2
4	3·065247584	3
5	4·959674775	5
6	8·024922359	8
7	12·98459713	13
8	21·00951949	21
9	33·99411663	34
10	55·00363612	55

Notice how, as n gets larger, the value of Phi^n/5 is almost an integer.

How many digits are there in Fib(n)?

Now you have enough information to answer the question:

How many digits has F(1000)?

Computing LOG (Phi^{1000} /(5)) is the same as computing
1000*LOG(Phi) - (LOG (5)) = 1000*LOG Phi - (LOG 5)/2.

So 1+the whole number part of your answer is the number of digits in F(1000).

In fact, you can find the first few digits by using the rest of the LOG answer, but I'll leave that for you to figure out, giving you the hint that the "opposite" (the inverse) function to LOG(n) is 10^n.

Calculating the next Fibonacci number directly

Suppose we have evaluated Fib(100) and we want to know the next value: Fib(101). Do we have to use Binet's formula again? Well we could do, of course, but here is a short cut.

There is also a formula that, given one Fibonacci number, returns the *next Fibonacci number* directly, calculating it in terms only of the previous value (i.e. not needing the value before as well).

If x is the value of F(n) then

F(n+1) = floor({x+1+[5 x^2]}/2)

The "floor" function floor(a) means "the next integer below a or a itself if a is an integer". For positive values, it means "rub out anything after the decimal point". The name comes from the picture of a building with floors at levels 0, 1, 2, etc (say 10 meters tall) and also some below ground labeled -1, -2, -3, etc. If we now hold an object at height "a" and let go, what "floor" will it land on?

floor(2·5)= 2   floor(2)= 2   floor(2·99)= 2   floor(2·00001)= 2

floor(-2·5)=-3   floor(-2)=-2   floor(-2·99)=-3   floor(-2·00001)=-3

Here's an example of the "next Fibonacci" formula using a small value of n to check it works:

Since F(5)=5 then F(6)=floor((5+1+sqrt(5x25))/2)
=floor((6 +sqrt(125))/2)
=floor((6 + 11·180)/2)
=floor(8·59)
=8

which is correct!

Here are two more examples:

You can easily evaluate F(0) and F(1) by this formula and see that they give 0 and 1 respectively. Then, if you are familiar with **proof by induction** you can show that, supposing the formula is true for F(n-1) and F(n) then it *must* also be true for F(n+1) by showing that adding the formula's expressions for F(n) and F(n-1) gives the formula's expression for F(n+1).

Other ways of proving it involve results about **recurrence relations** and how to solve them, which are very like solving differential equations, except that they deal with integer values not real number values.

Define multiplication on ordered pairs such as:

(A,B) (C,D) = (A C + A D + B C, A C + B D).

This is just (A X + B) * (C X + D) mod X^2 - X - 1, and so is associative, etc. We note (A,B) (1,0) = (A + B, A), which is the Fibonacci iteration. Thus, (1,0)^N = (FIB(N), FIB(N-1)), which can be computed in log N steps by repeated squaring, for instance. FIB(15) is best computed using N = 16, thus pushing the minimal binary addition chain counterexample to 30.

By the last formula:

(1,0)^-1 = (FIB(-1),FIB(-2)) = (1, -1)

which, as a multiplier, *backs up* one Fibonacci step (further complicating the addition chain question). Observing that
(1,0)^0 = (FIB(0), FIB(-1)) = (0,1)
= the (multiplicative) identity, equate it with scalar 1. Define addition and scalar multiplication as with ordinary vectors.
(A,B)^-1 = (-A, A + B) / (B^2 + A B - A^2),
so we can compute rational functions when the denominator isn't zero. Now, by using power series and Newton's method, we can compute fractional Fibonaccis, and even *e*^(X,Y) and log (X,Y). If we start with

(1,0) and square iteratively, the ratio will converge to the larger root of $x^2 - x - 1$ (= the golden ratio) about as rapidly as with Newton's method.

This method generalizes for other polynomial roots, being an improvement on the method of Bernoulli and Whittaker. For the general second order recurrence:

$$F(N+1) = X\, F(N) + Y\, F(N-1)$$

we have the multiplication rule:

$$(A,B)\,(C,D) = (A\,D + B\,C + X\,A\,C,\; B\,D + Y\,A\,C).$$

Inverse:

$$(A,B)^{-1} = (-A,\; X\,A + B) / (B^2 + X\,A\,B - Y\,A^2).$$

Two for the price of one:

$$(F(1),\; Y\,F(0))\,(1,0)^N = (F(N+1),\; Y\,F(N))).$$

Binet's Formula for negative n?

$$\textbf{Fib(n)} = \{\, \textbf{Phi}^{n} - (-\textbf{phi})^{n} \,\}/5$$

Here Phi=1·6180339... and phi = 1/Phi = Phi-1 = (sqrt(5)-1)/2 = 0·6180339... .

We only used this formula for **positive whole values of n** and found - surprisingly - it only gives integer results. Well perhaps it was not so surprising really since the formula is supposed to be define the Fibonacci numbers which *are* integers; but it *is* surprising in that this formula involves the square root of 5, Phi and phi which are all *irrational numbers* i.e. cannot be expressed exactly as the ratio of two whole numbers.

Suppose we try *negative* whole numbers for n in Binet's formula. The formula extends the definition of the Fibonacci numbers F(n) to negative n.

In fact, if we try to extend the Fibonacci series backwards, still keeping to the rule that a Fibonacci number is the sum of the two numbers on its LEFT, we get the following:

n :	... -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 ...
Fib(n):	... -8 5 -3 2 -1 1 0 1 1 2 3 5 8 ...

and this is consistent with Binet's formula for negative whole values of n.

So we can think of Fib(n) being defined for *all* integer values of n, both positive and negative and the Fibonacci series extending infinitely far in both the positive and negative directions.